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Boltzmann equation: old and new

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一、Introduction

The Boltzmann equation is one of the most fundamental equation of the kinetic theory, the unknown function f(t,x,v), which stands for the mass density function of gas particles having position x, velocity v, and time t . It satisfies a differential equation:

$$\frac{df}{dt} + v \cdot \nabla_x f = Q(f,f).$$

The left hand side of the equation represents the particle transport with the velocity v, and the right hand side of the equation represents the interaction between two colliding particles, it includes hard sphere case, hard potential case and soft potential case. However, if we consider inverse power force between particles, there will be some angular singularity in the collision operator. To avoid this mathematical difficulty, it was Grad's idea to cut the singularity. We will refer these cases as Grad's angular cut-off potential. The Boltzmann equation satisfies the conservation of mass, momentum and energy, and non-increasing of entropy, those physical properties will lead us get equilibrium state of the solution.

二、Mathematical theory for Boltzmann equation

Concerning the mathematical theory of the Boltzmann equation, it is nature to consider the solution near equilibrium. On the other hand, the nonlinear part usually decays faster than the linear part, so it is reasonable to drop the nonlinear part. We get the so called linearized Boltzmann equation after dropping the nonlinear part. For the study of the linearized Boltzmann equation, we can trace back to 1912, Hilbert [4] got the fluid structure of the linearized Boltzmann equation by using the expansion of the solution around a small Kundsen number, this expansion is the so called Hilbert expansion. We need to recognize that this kind of expansion is formal analysis and without mathematical proof. Concerning the mathematical structure of the linearized collision operator (spectrum structure), we do not have any result until Grad's paper in 1963. Grad got the explicit spectrum structure of the linearized collision operator for cutoff hard potentials case. After this result, the research of the Boltzmann equation improves very fast. For instance (only list the results relative to the author): Ukai (1974) [6] Perturbative solutions of the full inhomogeneous Boltzmann equation, based on the spectral theory of the linearized equation; Yan Guo (2002) [3] First of a series of works using energy methods to work out robust perturbative theories of the Boltzmann equation and other kinetic models; Liu-Yu (2004) [5] First works on the pointwise stability and Green function in the Boltzmann theory, and shock wave analysis; Gualdani-Mischle-Mouhot(2013) [2] Optimal rates of convergence for the Boltzmann equation in non-symmetric form.

三、Pointwise estimate

In general, the fundamental tools of the partial differential equation are energy estimate and Sobolev inequality. However, these kinds of methods will basically get the global norm estimate, but not localization estimate, i.e., the pointwise estimate. The first pointwise estimate of the linearized Boltzmann equation is hard sphere case, it was constructed by Tai-Ping Liu and Shih-Hsien Yu in 2004 [5], in their result, the solution of the linearized Boltzmann equation can be decomposed into the fluid part, the kinetic part and the remainder part. The crucial step of the construction is the so called "Mixture lemma", this lemma will transfer the velocity regularity to the space regularity. The original proof of the Mixture lemma need the explicit representation of the solution, this restriction can be replaced by the abstract method constructed by the author [7] (only need commutator analysis and energy estimate). Recently, the author (joint with Haitao Wang and Yu-Chu Lin) is able to construct the pointwise structure of the linearized Boltzmann equation for hard and soft potentials (need suitable velocity weight on the initial condition). We expect our ideas can apply to other important kinetic equations, for instance Landau equation; this will give more precise understanding of the kinetic theory.

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